Lesson 9. Markov Chains – Time-Independent Performance Measures

1 Overview

- Previous lesson: performance measures that depend on the number of time steps, for example

$$p_{ij}^{(n)} = \Pr\{S_n = j \mid S_0 = i\} \qquad\qquad p_j^{(n)} = \Pr\{S_n = j\}$$

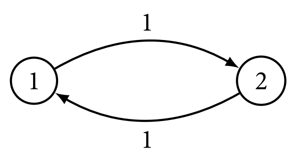
- This lesson: what happens in the **long run**, i.e. as $n \to \infty$? In particular, what is the **limiting probability**

$$p_{ij}^{(\infty)} = \lim_{n\to\infty} p_{ij}^{(n)} \quad ?$$

  - Note: the textbook uses $\vec{p}_{ij}$ instead of $p_{ij}^{(\infty)}$

2 Periodic and aperiodic states

- Consider the following two-state Markov chain:



- The n-step transition probability between state 1 and itself is:

- Now consider a Markov chain with state space $\mathcal{M} = \{1, \dots, m\}$
- A state $i \in \mathcal{M}$ is **periodic** with period δ (δ is a positive integer) if

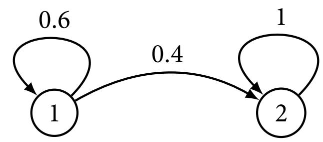
$$p_{ii}^{(n)} \begin{cases} > 0 & \text{if } n = \delta, 2\delta, 3\delta, \dots \\ = 0 & \text{otherwise} \end{cases}$$

  and therefore $p_{ii}^{(\infty)} = \lim_{n\to\infty} p_{ii}^{(n)}$ does not exist

- A state $i \in \mathcal{M}$ is **aperiodic** if it is not periodic
- In this class, we do not consider $p_{ij}^{(\infty)}$ for periodic states j

3 Transient and recurrent states

- Consider the following two-state Markov chain:

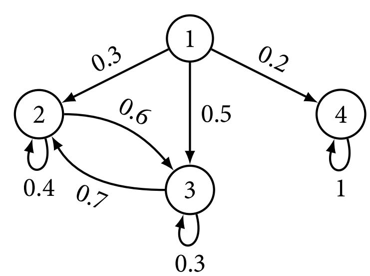


- The limiting probability between state 1 and itself is:

- In other words, the process eventually leaves state 1 and never returns
- A state $i \in \mathcal{M}$ is **transient** if $p_{ii}^{(\infty)} = 0$
  - The process will eventually leave state i and never return
- A state $i \in \mathcal{M}$ is **recurrent** if $p_{ii}^{(\infty)} > 0$
  - The process is guaranteed to return to state i over and over again, given that it reaches state i at some time

Example 1.

An autonomous UAV has been programmed to move between four regions to perform surveillance. The UAV is currently located in region 1, and moves between regions 1, 2, 3, and 4 according to a Markov chain with the transition-probability diagram on the right. Can you guess which states are transient and which states are recurrent?



- A subset of states $\mathcal{R}$ is **irreducible** if
  - $\mathcal{R}$ forms a self-contained Markov chain
  - no proper subset of $\mathcal{R}$ also forms a Markov chain
- Otherwise the subset $\mathcal{R}$ is **reducible**
- To find transient and recurrent states of a Markov chain:
  1. Find all irreducible proper subsets of the state space
     - If there are no such subsets, the entire state space is irreducible
  2. All states in an irreducible set are recurrent
  3. All states not in an irreducible set are transient

4 Steady-state and absorption probabilities

- Based on how the states are classified, we can compute the limiting probabilities $p_{ij}^{(\infty)}$
- **Case 1.** State j is transient.
  - $p_{ij}^{(\infty)} =$
  - Why? State j is transient $\Rightarrow$ will eventually leave state j and never return
- **Case 2.** States i and j are in <u>different</u> irreducible sets of states.
  - $p_{ij}^{(\infty)} =$
  - Why? State i is one self-contained Markov chain, state j is in another
- **Case 3.** States i and j are in <u>the same</u> irreducible set of states $\mathcal{R}$.
  - $p_{ij}^{(\infty)} = \pi_j$ for some $\pi_j > 0$ — note that $p_{ij}^{(\infty)}$ in this case <u>does not depend on i</u>!
  - We can compute π_j by solving the following system of linear equations:

    where
    $\pi_{\mathcal{R}}$ = vector of π_j for $j \in \mathcal{R}$
    $\mathbf{0}$ = vector of zeros
    $\mathbf{1}$ = vector of ones
  - The π_j are called **steady-state probabilities**
  - Interpretation: given that the process reaches the irreducible set containing state j, π_j is
    - the probability of finding the process in state j after a long time, or
    - the long-run fraction of time that the process spends in state j
  - Note that by construction, the steady-state probabilities add up to 1

Example 2. Consider the UAV example again. Suppose the UAV reaches region 2 at some point. What is the long-run fraction of time that the UAV spends in region 2? Region 3?

- **Case 4.** State i is transient and state j is an **absorbing** state
  (i.e. state j is the only state in an irreducible set of states $\mathcal{R} = \{j\}$)
  - $p_{ij}^{(\infty)} = \alpha_{ij}$ for some $\alpha_{ij} \geq 0$
  - Let $\mathcal{T}$ be the set of transient states
  - Let $\alpha_{\mathcal{T}\mathcal{R}}$ be the vector whose elements are α_{ij} for $i \in \mathcal{T}$ (remember $\mathcal{R} = \{j\}$)
  - We can find the α_{ij}'s using:
  - The α_{ij} are called **absorption probabilities**
    - What is the probability that the process is ultimately "absorbed" at state j?

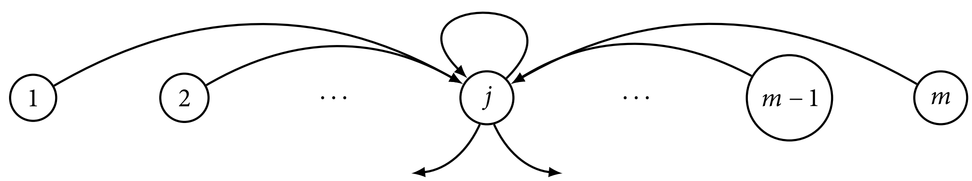
Example 3. Consider the UAV example again. What is the probability that the UAV is in region 4?

- We can find the probability that the process is ultimately absorbed into an irreducible set of states $\mathcal{R}$ (possibly with more than 1 state) by lumping the states in $\mathcal{R}$ into a "super state" and then applying the concepts above (I'll let you think about this.)

5 Why are the steady-state probabilities computed this way?

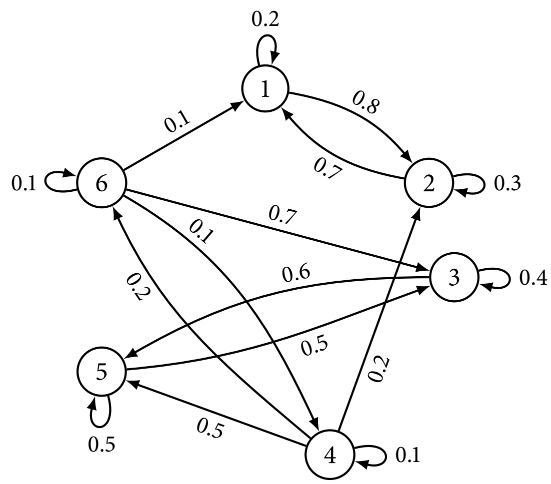
- Some details in Nelson, pp. 153-154
- Intuition: in steady state, we have that

$$\text{frequency of being in state } j = \text{frequency of transitions into state } j$$



6 Exercises

Problem 1. An autonomous UAV has been programmed to move between six regions to perform surveillance. The movements of the UAV follow a Markov chain with 6 states (1 for each region), and the following transition probability diagram:



a. There are two irreducible sets of states: $\{1,2\}$ and $\{3,5\}$. Briefly explain why these sets are irreducible.
b. Which states are transient? Which states are recurrent? Briefly explain.
c. Suppose the UAV starts in region 1. What is the long-run fraction of time that the UAV spends in region 1?
d. What is the probability that the UAV is absorbed into states 3 or 5, given that it starts in region 4?